Manifold Calculus and the \( h \)-principle
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Abstract

in this project we use manifold calculus and \( h \)-principle to study a problem in symplectic geometry.

\[ \text{manifold calculus + \( h \)-principle} \land \text{symplectic geometry} \]

1 Motivation
a symplectic manifold is a pair \((N, \omega)\) where \(N\) is a smooth manifold of dimension \(2m\) and \(\omega \in \Omega^2_{\text{DeRham}}(N)\) satisfying

1. \(d\omega = 0\)
2. \(\omega^m\) is nowhere vanishing

\(\omega\) defines an isomorphism \(\omega : TM \rightarrow T^*M\)

Example 1.1. \(N = \mathbb{R}^{2m}\)
\(\omega = \sum_{i=1}^{m} dp_i \wedge dq_i\)

Example 1.2. \(N = T^*M\)
(implicitly) \(= \sum_{i=1}^{m} dp_i \wedge dq_i\)

where \(q_i\) are the base coordinates and \(p_i\) are the tangent coordinates.

in classical mechanics, \(T^*M\) is the phase space, \(q_i\) are the position coordinates and \(p_i\) are the momentum coordinates.

Darboux’s theorem says that every symplectic manifold is locally symplectomorphic to \(\mathbb{R}^{2m}\) with the standard symplectic form, and so symplectic manifolds have no local invariants but they do have very non-trivial global invariants; see [CdS01]. many global invariants are constructed out of Lagrangian submanifolds.

a Lagrangian submanifold is a submanifold \(M \subseteq N\) of dimension \(m\) such that \(\omega|_M \equiv 0\)

\(\text{Emb}_{\text{Lag}}(M, N)\) is the space of Lagrangian embeddings \(M \hookrightarrow N\)

The Nearby Lagrangian Conjecture (still open) due to Arnol’d has been a guiding question for several recent advances in symplectic geometry. The current state of the art results about the Nearby Lagrangian Conjecture rely on a combination of homotopy theoretic and Floer theoretic techniques.

Conjecture 1.3 (Arnold’s Nearby Lagrangian Conjecture). If \(L\) and \(M\) are simply connected closed manifolds of dim \(m\), then the space

\(\text{Emb}_{\text{Lag}}(L, T^*M)\) is \[
\begin{cases} 
\text{contractible} & \text{if } L \cong M, \\
\text{empty} & \text{otherwise.}
\end{cases}
\]

the goal of this project is to apply homotopy theoretical methods (manifold calculus) to study \(\text{Emb}_{\text{Lag}}(M, N)\).

\(^1\)basic example is the position space inside a phase space, in classical mechanics.
2 Statement of the Theorem

The **Spaces**-enriched category of smooth manifolds of a fixed dimension $m$, with morphisms being the embedding spaces.

The full subcategory of **Man** consisting of manifolds which are diffeomorphic to disjoint union of $\mathbb{R}^m$.

Space valued presheaves on **Man**, functors $\text{Man}^{op} \to \text{Spaces}$.

Right derived Kan extension of $F \in \text{Psh}(\text{Man})$ along the inclusion $\text{Disc}^{op}_\infty \hookrightarrow \text{Man}^{op}$.

We say that $T_\infty F$ is the **analytic approximation** of $F$.

See [Wei99], [BdBW12], [MV15].

**Theorem 2.1 (N., [Nak17]).** Let $(N, \omega)$ be a symplectic manifold with dimensions $n = 2m$. For $m > 2$ there is a homotopy equivalence

$$T_\infty \text{Emb}_{\text{Lag}}(\cdot, N) \simeq \text{Emb}_{\text{TR}}(\cdot, N)$$

where

$\text{Emb}_{\text{Lag}}(\cdot, N)$ is the space of Lagrangian embeddings and
$\text{Emb}_{\text{TR}}(\cdot, N)$ is the space of totally real embeddings.

2.1 Totally Real Submanifolds
a compatible almost complex structure on \((N, \omega)\) is a vector space automorphism \(J : TN \to TN\) with \(J^2 = -1\) such that

1. \(\omega(J -, -)\) defines a Riemannian metric on \(N\),
2. \(\omega(J -, J -) = \omega(-, -)\).

there exists a unique compatible almost complex structure up to homotopy, we’ll assume to have chosen one

a totally real manifold is a submanifold \(M \subseteq N\) of dimension \(m\) such that \(TM \oplus J(TM) \cong TN\big|_M\) \(^2\)

\(\operatorname{Emb}_{\text{TR}}(M, N)\) is the space of totally real embeddings \(M \hookrightarrow N\) by compatibility, every Lagrangian submanifold is also a totally real submanifold,

\(\operatorname{Emb}_{\text{Lag}}(M, N) \subseteq \operatorname{Emb}_{\text{TR}}(M, N)\)

## 3 Proof

following are the steps to prove 2.1:

1. \(\operatorname{Imm}_{\text{Lag}}(M, N) \overset{\sim}{\rightarrow} \operatorname{Imm}_{\text{TR}}(M, N)\), see [EM01].
   - immersions are linear, in the sense of manifold calculus
   - so the homotopy type is completely determined by the tangential data
   - on the level of tangent spaces the result follows by the homotopy equivalence \(U(m)/O(m) \overset{\sim}{\rightarrow} GL(m, \mathbb{C})/GL(m, \mathbb{R})\), see [Arn67].

2. \(T_\infty \operatorname{Emb}_{\text{Lag}}(M, N) \overset{\sim}{\rightarrow} T_\infty \operatorname{Emb}_{\text{TR}}(M, N)\)

\(^2\)the basic example is \(\mathbb{R}^m \subseteq \mathbb{C}^m\).
• this is because of the following homotopy pullback diagram

\[
\begin{array}{c}
T_\infty \Emb_{\text{Lag}}(M, N) \\
\downarrow \\
\Imm_{\text{Lag}}(M, N)
\end{array} \longrightarrow 
\begin{array}{c}
T_\infty \Emb_{\text{TR}}(M, N) \\
\downarrow \\
\Imm_{\text{TR}}(M, N)
\end{array}
\]

3. \(\Emb_{\text{TR}}(M, N) \isomorph T_\infty \Emb_{\text{TR}}(M, N)\)

• this last step in the proof uses Gromov’s \(h\)-principle for directed embeddings. See [Gro86], [Spr98].

4 Gromov’s \(h\)-principle

\(\Gr_m(N) \rightarrow N\) is the \(m\)-plane Grassmannian bundle over \(N\)

\(\text{Lag} \subseteq \Gr_m(N)\) subbundle corresponding to the Lagrangian subspaces of \(TN\)

\(\text{TR} \subseteq \Gr_m(N)\) subbundle corresponding to the totally real subspaces of \(TN\)

by compatibility, \(\text{Lag} \subseteq \text{TR}\).

an embedding \(e : M \hookrightarrow N\) induces a map

\(\Gr_m(e) : M \rightarrow \Gr_m(TN)\)

when \(e\) is a totally real (resp. Lagrangian) embedding the image of \(\Gr_m(e)\) lies in \(\text{TR}\) (resp. \(\text{Lag}\)).

Gromov’s idea (originally by Smale) was to consider the space of \textbf{formal totally real sections} consisting of triples \((e, s, \gamma)\)

\(e\) an embedding \(M \rightarrow N\)

\(s\) a section of \(\text{TR}\) over \(e(M)\)

\(\gamma\) a path of sections of \(\Gr_m(TN)\) over \(e(M)\) from \(De(TM)\) to \(s\)

a formal section \((e, s, \gamma)\) is called \textbf{holonomic} if \(s = De(TM)\) and \(\gamma\) is the constant path.
Gromov showed using his technique of convex integration that the space of formal totally real sections is homotopy equivalent to the space of holonomic totally real sections.

\[
\{ \text{formal totally real sections} \} \simeq \{ \text{holonomic totally real sections} \}
\]

this is false for lagrangian sections. this is because the space $\text{TR}$ is large but $\text{Lag}$ is small. more specifically, the complement of $\text{TR}$ inside $\text{Gr}_m(N)$ is a variety of codimension $\geq 2$ (it is the complement of a thin singularity).

5 Future Work

1. how to incorporate Floer theoretic data in the Goodwillie-Weiss tower?
2. using manifold calculus to study foliations (which are studying using $h$-principle)
3. extending the above techniques to manifolds with a group action (current work in progress).

References


